3.6. Linear optimization, simplex algorithm

Linear optimization (or linear programming): subject of applied mathematics with a well-founded theory,

The state Brandenburg would like to support the sustainable land use. For this purpose the state will use two programs: the exensification programm (EX) and the organic farming support program (OFS). Concerning the sustainable land use, the extensification program has a target contribution of 3 and the organic farming support program has a target contribution of 5. Six million euros will be available for both programs.
The implementation of the programs creates administrative expenses: The implementation of EX will need one civil servant per one million euro, the implementation of OFS will need two civil servant per 1 million euro, whereas seven civil servants will be available for both programs.
In addition, the implementation of the programs will need additional farm workload: The implementation of EX will need additional farm workload of three workers per one million euro, the implementation of OFS will need nine workers per one million euro, whereas the total additional workload should be lower than 27 workers.
To what extent EX and OFS should be supported?

Variables: x1 Mio. € for OFS
x2 Mio. € for EX.

The mathematic modelling produces the following linear optimization problem:

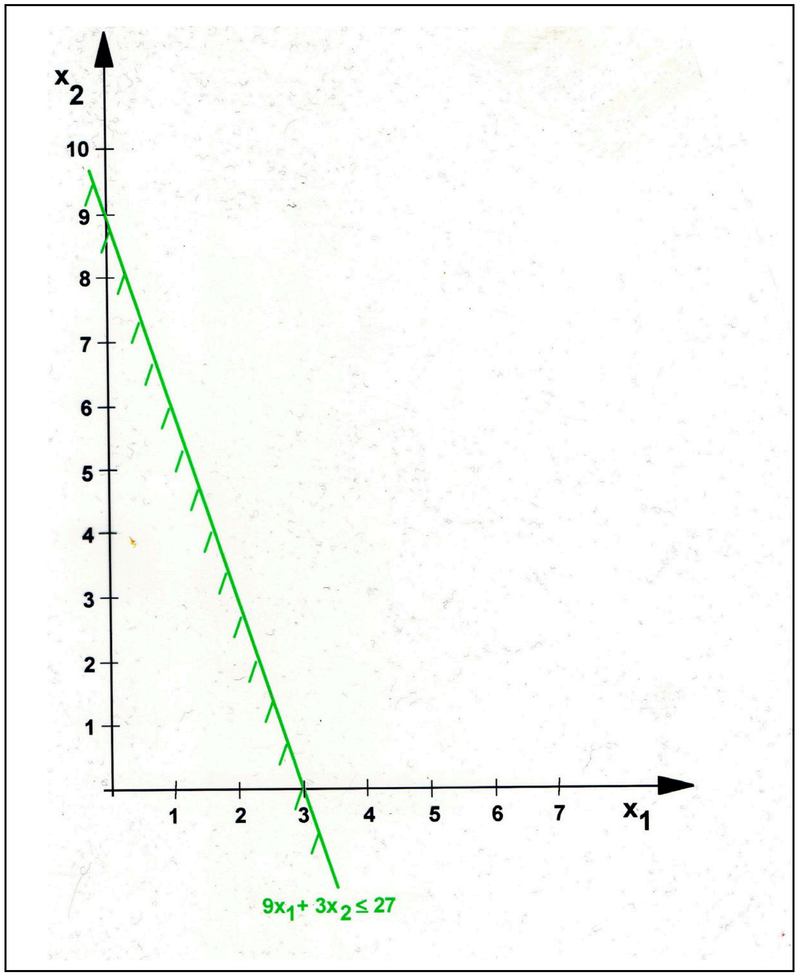
$$\max\left\{5x_1 + 3x_2 \left| \begin{array}{c} 9x_1 + 3x_2 \le 27 \\ 2x_1 + x_2 \le 7 \\ x_1 + x_2 \le 6 \end{array} \right., x_1 \ge 0,\ x_2 \ge 0\right\}$$

with the objective function(OF) and the constraints;

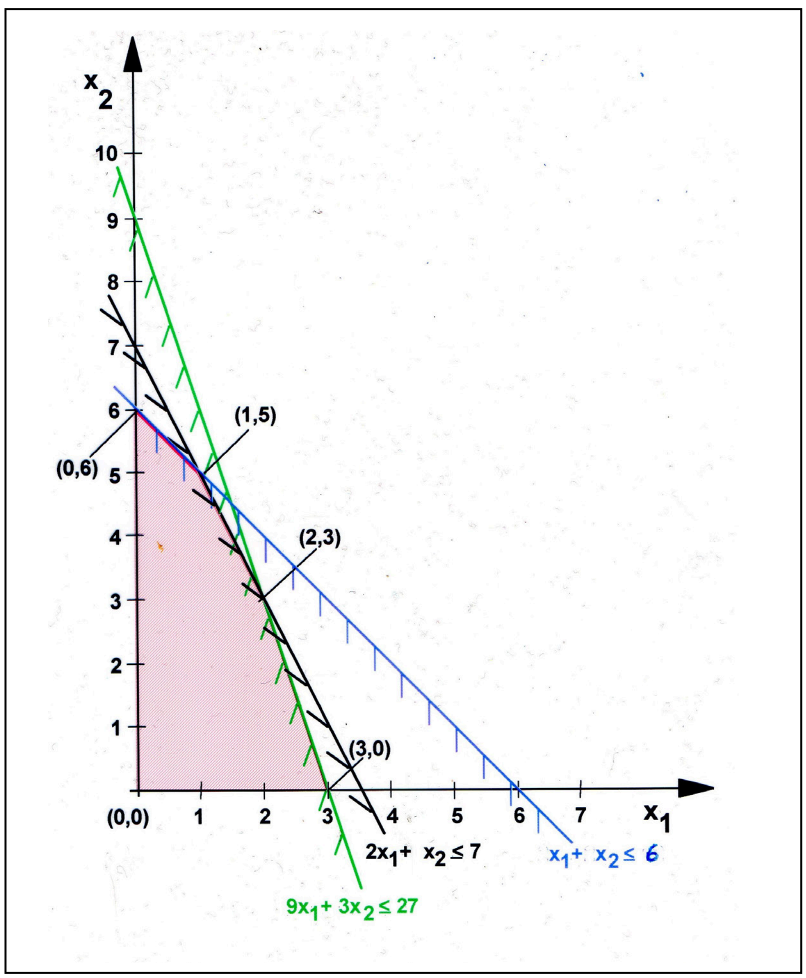
- considering two variables it is possible to solve the problem and to represent it geometrically
- but the geometric solution is inexact and only applicable to a limited extent (2 variables)

Geometric solution

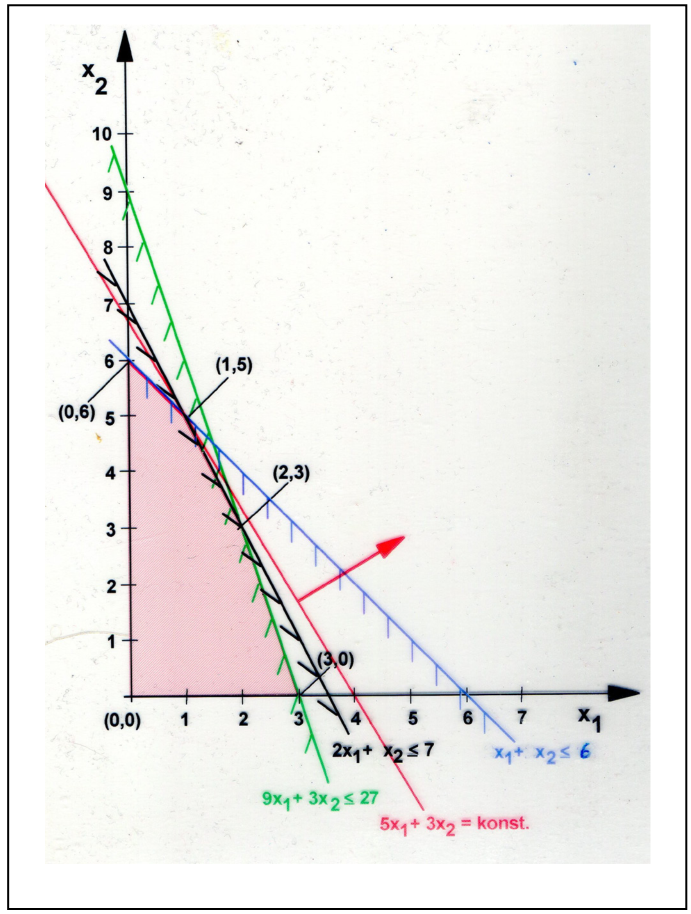
First halfplane



Set of feasible solutions



Optimum solution in point (1, 5) as point of contact (for a parallel shift) of the objective function and the set of feasible solution



Arithmetic solution

Simplex algorithm (G. B. Dantzig, 1948/49),
For that we transfer three inequalities by introducing slack variables x_3, x_4, x_5 (all ≥ 0) in equations:

$$\begin{aligned} 9x_1 + 3x_2 + x_3 &= 27 \\ 2x_1 + x_2 + x_4 &= 7 \\ x_1 + x_2 + x_5 &= 6 \end{aligned}$$

, $OF: 5x_1 + 3x_2 + 0x_3 + 0x_4 + 0x_5$,

Resolving into slack variables.:

$$\begin{aligned} x_3 &= 27 - (9x_1 + 3x_2) \\ x_4 &= 7 - (2x_1 + x_2) \\ x_5 &= 6 - (x_1 + x_2) \end{aligned}$$

Basic variables (BV): x_3, x_4, x_5

Non basic variables (NBV): x_1, x_2

creates the following Simplex tableau:

			5	3
			x_1	x_2
0	x_3	27	9	3
0	x_4	7	2	1
0	x_5	6	1	1
		0	-5	-3

BV: x_3 = 27, x_4 = 7, x_5 = 6

NBV: x_1 = 0, x_2 = 0

We get a new simplex tableau,this is similar to ETB, but we need to choose the pivot element so that the objective function improves.

Rules:

<u>Pivot column:</u> min {-5, -3 } = -5

(see last row: *Characteristic row*)

If the definite value is ≥ 0 , the related vector is a optimal solution.

<u>Pivot row:</u>

$$\min\left\{\tfrac{27}{9},\tfrac{7}{2},\tfrac{6}{1}\right\}=\tfrac{27}{9}$$

By minimizing we consider only those fractions having a positive denominator. If every values of the pivot column are negative, there is no optimal solution.

			0	3
			x_3	x_2
5	x_1	3	1/9	1/3
0	x_4	1	-2/9	1/3
0	x_5	3	-1/9	2/3
		15	5/9	-4/3

BV: x_1 = 3, x_4 = 1, x_5 = 3

NBV: x_3 = 0, x_2 = 0

PC: $\min\left\{\frac{5}{9},\frac{-4}{3}\right\}=\frac{-4}{3}$

PR: $\min\left\{\frac{3}{\frac{1}{3}},\frac{1}{\frac{1}{3}},\frac{3}{\frac{2}{3}}\right\}=\frac{1}{\frac{1}{3}}$

			0	0
			x_3	x_4
5	x_1	2	1/3	-1
3	x_2	3	-2/3	3
0	x_5	1	1/3	-2
		19	-1/3	4

BV: x_1 = 2, x_2 = 3, x_5 = 1

NBV: x_3 = 0, x_4 = 0

PC: $\min\left\{\frac{-1}{3},4\right\}=\frac{-1}{3}$

PR: $\min\left\{\frac{2}{\frac{1}{3}},\frac{1}{\frac{1}{3}}\right\}=\frac{1}{\frac{1}{3}}$

			0	0
			x_5	x_4
5	x_1	1	-1	1
3	x_2	5	2	-1
0	x_3	3	3	-6
		20	1	2

BV: x_1 = 1, x_2 = 5, x_3 = 3 (not required workers) NBV: x_5 = 0, x_4 = 0

$$\min\{1,2\}=1\geq 0$$

$\Rightarrow$ We got an optimal solution!

Linear optimization problem:

Maximizing or minimizing a linear function under the constraints of a system of linear equations and inequalities and non negative variables.

max $\{\underline{c}^T\underline{x} | \underline{x} \epsilon B\}$

$$B = \left\{ \underline{x} \in R^n \left| \begin{array}{lll} \sum_{j=1}^{n} a_{ij}x_j = b_i & , & i \in E \\ \sum_{j=1}^{n} a_{ij}x_j \leq b_i & , & i \in LE \quad , \quad \underline{x} \geq \underline{0} \\ \sum_{j=1}^{n} a_{ij}x_j \geq b_i & , & i \in GE \end{array} \right. \right\}$$

E, LE , GE are index sets

B is a convex polyhedron: intersection of a finite number of closed half-spaces.

We have:

min $\{\underline{c}^T\underline{x} | \underline{x} \epsilon B\}$ = - max$\{-\underline{c}^T\underline{x} | \underline{x} \epsilon B\}$, that means a linear minimizing problem can be reduced to a maximizing problem.

(1) $B \neq \emptyset$ there is

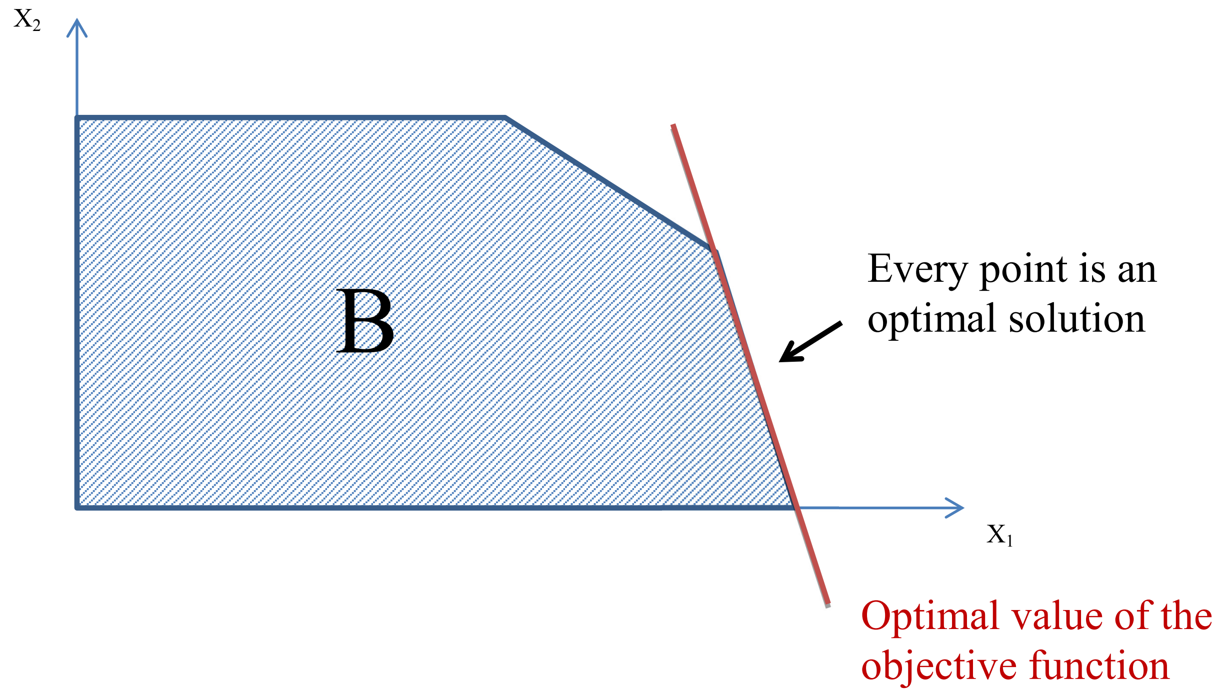
a) exactly one optimal solution (a corner point of the convex polyhedron B),see above ,

b) an infinite number of optimal solutions (an edge or a side of the convex polyhedron B)

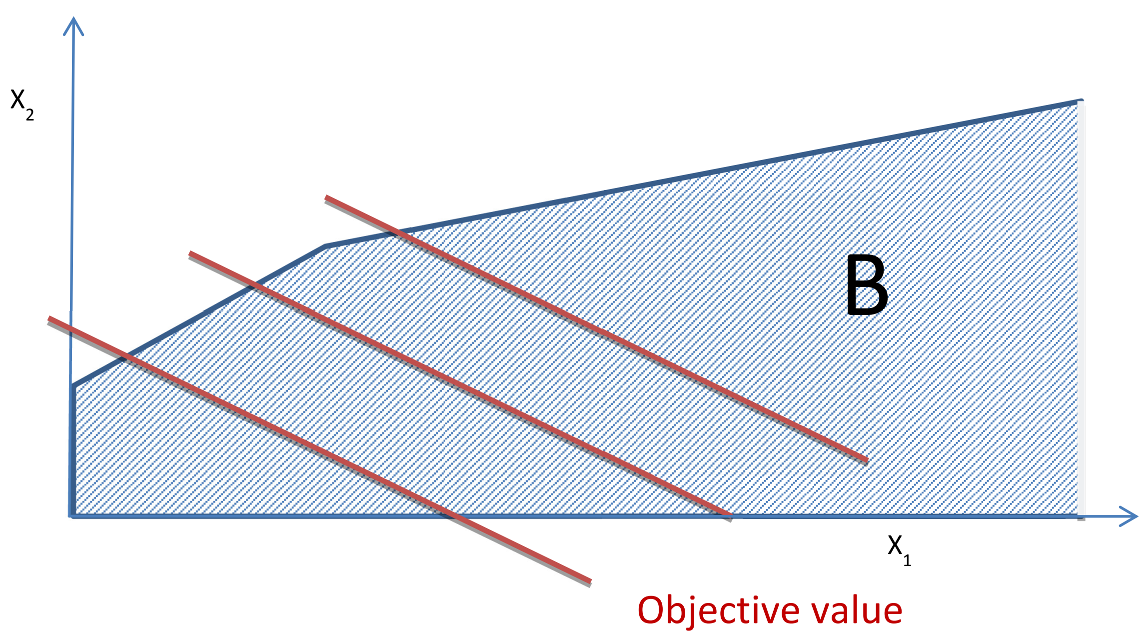
c) no optimal solution.

(2) $B = \emptyset$

b)


X2
B
Every point is an optimal solution
X1
Optimal value of the objective function

c)


X2
B
X1
Objective value

(2) $B = \emptyset$ For example: three half spaces which do not overlap all together.